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Sub-aging in a domain growth model

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Abstract. We study analytically the aging dynamics of the $O(n)$ model in the limit of $n \to \infty$, with conserved and with non-conserved order parameter. While in the non-conserved dynamics, the autocorrelation function scales in the usual way $C(t,t_w) = \mathcal{C}(t/t_w)$, in the case of a conserved order parameter, 'multiscaling' manifests itself in the form $C(t,t_w) = \mathcal{C}(h(t)/h(t_w))$, with a relaxation time growing more slowly than the age of the system (sub-aging), and $h(t)$ a function growing faster than any length scale of the problem. In both cases, the effective temperature associated to the violation of the fluctuation theorem tends to infinity in the asymptotic limit of large waiting times.

PACS. 05.70.Ln Nonequilibrium and irreversible thermodynamics – 75.40.Gb Dynamic properties (dynamic susceptibility, spin waves, spin diffusion, dynamic scaling, etc.) – 82.20.Mj Nonequilibrium kinetics

1 Introduction

Aging of glassy systems is now well understood, at least from a qualitative point of view [1], and different approaches have been used to understand such a behaviour. One of them is the interpretation of aging in terms of a coarsening process. The picture is the following: consider for instance an Ising ferromagnet, which is quenched at time $t = 0$ below its critical temperature. When t increases, two types of domains emerge, with up and down spins. In the thermodynamic limit, equilibrium is never reached. At late times, domains have reached a typical size $L(t)$. It is thus natural to assume scaling laws for the different quantities of interest [2]. For instance, one can try the ansatz $S(\mathbf{k},t) \sim L^d g(kL)$ for the structure factor (in a d-dimensional space), or $C(t,t_w) \sim F(L(t)/L(t_w))$ for the two-time autocorrelation function, where g and F are scaling functions. The growth law $L(t)$ determines then all the properties of the system. As an example, the droplet model for spin glasses [3] assumes a logarithmic growth, leading to $C(t,t_w) = F(\ln t/\ln t_w)$. If the growth law is given by $L(t) \sim t^\alpha$, like *e.g.* in a spinodal decomposition, one gets $C(t,t_w) = F(t/t_w)$. This last behaviour is called 'simple aging' and has been analytically shown to hold within various non-random models [1,4].

Moreover, the above functional form for the correlation function is also found analytically in some mean-field models of spin glasses, which give the general form for the correlation functions in the aging regime $C(t,t_w) = \mathcal{C}(h(t)/h(t_w))$, with h and $\mathcal{C}$ two scaling functions [1] (valid in the two-time regime where both times are large, but with $1 < C < 0$). Although the notations are different, the functional form is the same as in coarsening processes, and it is then very natural to try to interpret the h-function as a relevant length scale for spin glasses, as was done for instance in reference [5].

From the experimental and numerical side, it is found that a simple aging behaviour describes the data well, in many different systems. This is interpreted by saying that the relaxation time $t_r(t_w)$ of the system scales as its age t_w: $t_r \propto t_w$. However, a more subtle effect may appear, since t_r very often *grows more slowly than* t_w. This effect has been called *sub-aging* [6]. In his pioneering experiments on polymer glasses, Struik [7] introduced the exponent μ from the relation $t_r \sim t_w^\mu$, with $\mu < 1$. Different values of μ have been reported: Struik used $\mu \sim 0.9$, experiments in spin glasses indicate that $\mu \sim 0.97$ [6], simulations of a structural glass were fitted using the value $\mu \sim 0.88$ [8], and recently, experiments on a gel gave $\mu \sim 0.9$ [9]. It can be checked (this point is discussed in detail in reference [6]) that the μ-exponent is equivalent to the following choice for the h-function: $h(t) = \exp(\frac{1}{1-\mu}(\frac{t}{t_0})^{1-\mu})$, and this equivalence holds in the two-time regime characterised by $t_w \to \infty$ and $t - t_w \sim t_w^\mu$. Another function, the 'enhanced power law' form $h(t) = \exp(\ln^a(t/t_0))$ with $a > 1$, has been *phenomenologically* introduced in the context of spin glasses [6], and the value $a = 2.2$ was used to fit experiments. This in turn gives the relation $t_r \sim t_w/\ln^{a-1}(t_w)$, valid in the regime $t_w \to \infty$ and $t - t_w \sim t_w/\ln^{a-1}(t_w)$. It is interesting to note that sub-aging is *not* found in numerical simulations of 3D spin glass models where the t/t_w-scaling seems to be a robust one [10].

However, the above sub-aging scaling forms *are not motivated by theoretical arguments*, since the mean-field spin glass models discussed above only predict

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the existence of $h(t)$, and its analytical computation remains at present an open problem. In this context, simple models where h can be computed are much needed, but there are only very few examples where sub-aging appears. Recently, a model exhibiting sub-aging has been proposed by Rinn *et al.* [11], who studied a slight variation of Bouchaud's trap model for aging. This has given a theoretical support to the use of an exponent μ, even if its physical origin remains somewhat unclear. A scaling approach to the diffusion of a point particle in a low dimensional space has been proposed in reference [12], and leads in some cases to a sub-aging behaviour which can be well described by an enhanced power law.

We consider in the present paper a well-studied model [4,13,14] for coarsening (the $O(n)$ model in the large-n limit) and show that it exhibits a sub-aging scaling in the autocorrelation function when the order parameter is conserved. Its origin is the *simultaneous presence in the system of two different length scales,* whose consequence is the breakdown of the simple scaling laws generally used in domain growth processes. In particular, no t/t_w-scaling is found, and the relaxation time grows as $t_r \sim t_w/\sqrt{\ln t_w}$ (sub-aging). The autocorrelation is shown to be well represented in the asymptotic regime by a function h which is an enhanced power law with $a = 3/2$, *i.e.* $h(t) = \exp((\ln x)^{3/2})$, which justify its use in fitting spin glass experiments. Interestingly enough, $h(t)$ can not be interpreted in our example as a length scale. We do not want to argue that the model is a realistic one for the aging of polymers or spin glasses, but rather to give a possible physical explanation (the role of length scales [15]) for the absence of the 'naive' t/t_w-scaling, and exhibit a simple example where the h-function can be computed and discussed in terms of length scales, which has not been done so far.

2 The O(N) model

This model is one of the few exactly solvable models for coarsening, as was shown first by Mazenko and Zannetti [13]. Coniglio and Zannetti [14] computed further the scaling properties of the structure factor during the domain growth process and pointed out the presence of the two mentioned length scales, and named 'multiscaling' the breakdown of the usual $S(\mathbf{k},t) \sim L^d g(kL)$. Bray and Humayun have shown, however, that this multiscaling was a peculiarity of the large-n limit, and proved that for a large but finite value of n, a 'normal scaling' was recovered [16]. On an other hand, this 'pathology' has been shown to appear as a relevant preasymptotic effect in different coarsening models [17], like for instance the kinetic Ising model. We shall make use of previous published results [13,14] to study the aging properties of the model, usually studied through the autocorrelation function, that have not been derived yet for the conserved order parameter case. We compute also the response functions in both cases.

The model is defined through the Hamiltonian [2]

$$H[\boldsymbol{\phi}] = \int \mathrm{d}^d\mathbf{x} \left(\frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\phi})^2 + \frac{1}{4n}(n - \boldsymbol{\phi}^2)^2 \right), \qquad (2.1)$$

where $\boldsymbol{\phi}(\mathbf{x},t)$ is a n-component vector field in a d-dimensional space. Two different dynamics may be associated to this model, depending on whether or not the order parameter is conserved. In the case of a non-conserved order parameter, the dynamics is given by the so-called time dependent Ginzburg-Landau equation

$$\frac{\partial \boldsymbol{\phi}(\mathbf{x},\mathbf{t})}{\partial t} = -\frac{\delta H}{\delta \boldsymbol{\phi}(\mathbf{x},\mathbf{t})}\,. \qquad (2.2)$$

For conserved fields, we add $-\nabla^2$ in front of the r.h.s to get the Cahn-Hilliard equation

$$\frac{\partial \boldsymbol{\phi}(\mathbf{x},\mathbf{t})}{\partial t} = \nabla^2 \left(\frac{\delta H}{\delta \boldsymbol{\phi}(\mathbf{x},\mathbf{t})} \right). \qquad (2.3)$$

The limit $n \to \infty$ allows to solve the dynamics in both cases, because the replacement $\boldsymbol{\phi}^2/n \to \langle \phi^2 \rangle$, where ϕ is one of the components of $\boldsymbol{\phi}$, becomes correct [13]. The problem may be studied at $T = 0$ (there is no noise terms in the dynamical equations), since in coarsening problems, temperature does not play an essential role, provided it is below the critical temperature. (The review paper [2] provides a longer discussion of that point.)

For the non-conserved dynamics, one has for the structure factor [2,13]

$$S(\mathbf{k},t) \equiv \frac{1}{V}\langle \phi(\mathbf{k},t)\phi(-\mathbf{k},t)\rangle = (8\pi t)^{d/2}\mathrm{e}^{-2k^2 t}. \qquad (2.4)$$

and for the autocorrelation function

$$C(t,t_w) \equiv \frac{1}{V}\int \mathrm{d}^d\mathbf{x}\, \langle \phi(\mathbf{x},t)\phi(\mathbf{x},t_w)\rangle = \left[\frac{2\sqrt{tt_w}}{t+t_w}\right]^{d/2}. \qquad (2.5)$$

Defining the length scale $L(t) \equiv t^{1/2}$ and the scaling variables $x \equiv kL(t)$, $\lambda_1 \equiv t/t_w$, it is possible to write

$$\begin{aligned} S(\mathbf{k},t) &= L^d g(x), \quad g(x) = (8\pi)^{d/2}\exp(-2x^2), \\ C(t,t_w) &= F_1(\lambda_1), \quad F_1(x) \equiv \left[\frac{2\sqrt{x}}{1+x}\right]^{d/2}. \end{aligned} \qquad (2.6)$$

These expressions give a specific example for the scaling scenario of the introduction. In particular, the autocorrelation function exhibits a simple aging behaviour. By using a power law $h(t) = t^\alpha$ [for instance $h(t) = L(t)$], the last equation can be recast in the general $C(t,t_w) = \mathcal{C}(h(t)/h(t_w))$ scaling form.

3 Conserved order parameter: sub-aging

The Cahn-Hilliard equation (2.3) associated to the Hamiltonian (2.1) may be solved in the large-n limit

to give [2,13,14]

$$S(\mathbf{k},t) \sim \left[L_1(t)^d\right]^{f(kL_2(t))}, \qquad (3.1)$$

where $f(x) \equiv 2x^2 - x^4$. In this expression, two characteristic length scales have been defined: $L_1(t) \equiv t^{1/4}$, and $L_2(t) \equiv (\frac{8t}{d\ln(t/t_0)})^{1/4}$; t_0 is an uninteresting constant. In the standard scaling form, $S(\mathbf{k},t) \sim L^d g(kL)$, the structure factor varies as L^d with a prefactor depending on the scaling variable kL, whereas for the multiscaling form (3.1), S varies as L_1^α, with an exponent α which depends continuously on the scaling variable kL_2. The two scalings are thus completely different.

Coniglio and Zannetti [14] have interpreted this multiscaling in terms of domains composed of sub-domains, each sub-domain growing at a different rate. The initial motivation for the present work was indeed to investigate the possible existence of a 'hierarchy' of time scales, similar to the one found in mean-field spin glass models ('ultrametricity in time') [1,11,18,19]. A different effect arises instead, since one gets for the autocorrelation function

$$C(t,t_w) \sim \frac{1}{(t+t_w)^{d/4}} \times \exp\left(\frac{d}{8}\frac{\left(\sqrt{t\ln(t/t_0)}+\sqrt{t_w\ln(t_w/t_0)}\right)^2}{t+t_w}\right). \qquad (3.2)$$

It is obvious from this expression that $C(t,t_w)$ cannot be written as a function of t/t_w only. The physical key ingredient for the absence of the usual scaling is the presence of two different length scales in the system.

We prove now analytically that equation (3.2) implies sub-aging. It has to be remarked first that when the time difference $\tau \equiv t - t_w$ is equal to t_w, one has

$$C(t_w+t_w,t_w) \underset{t_w\to\infty}{\sim} \frac{1}{t_w^{(3-2\sqrt{2})d/24}} \to 0. \qquad (3.3)$$

In the asymptotic limit of large waiting times, the relaxation of $C(t,t_w)$ is complete in times $\tau \ll t_w$. In that regime, one can show that

$$C(t,t_w) \underset{\tau\ll t_w}{\sim} \exp\left(-\frac{d\ln t_w}{64}\left(\frac{\tau}{t_w}\right)^2\right). \qquad (3.4)$$

Defining the scaling variable $\lambda_2 \equiv \tau\sqrt{\ln t_w}/t_w$, equation (3.2) can finally be rewritten

$$C(t,t_w) \sim F_2(\lambda_2); \quad F_2(x) \equiv \exp\left(-\frac{dx^2}{64}\right). \qquad (3.5)$$

The relaxation time grows hence as $t_r \sim t_w/\sqrt{\ln t_w}$, *i.e.* more slowly than t_w: *this is a sub-aging behaviour*. It is moreover possible to compute the function $h(t)$ discussed in the introduction. The scaling form $C(t,t_w) = \mathcal{C}\left(h(t)/h(t_w)\right)$ should be valid in the two-time regime where both times are large, but with a non-zero value of the correlation function. In the present case, this regime is characterized by $t_w \to \infty$, $\tau \sim t_w/\sqrt{\ln t_w}$. We have seen that a natural choice for $h(t)$ would be $L_1(t)$ or $L_2(t)$, *i.e.* a length scale, since it is a common interpretation. *This does not work*, and a more complicated form has to be found. It is simple algebra to prove that $h(t)$ is in fact an enhanced power law:

$$\mathcal{C}(x) = \exp\left(-\frac{d}{288}\ln^2(x)\right); \quad h(t) = \exp\left((\ln t)^{3/2}\right). \qquad (3.6)$$

The function h is neither L_1 nor L_2, but a combination of the two, and therefore does not have a direct physical interpretation: $h(t) \sim \exp\left((L_1/L_2)^6\right)$.

4 Response functions: infinite effective temperatures

It is also relevant to study the response functions for aging systems, since it is a major prediction of the dynamical mean-field theory for spin glasses that interesting informations are encoded in the susceptibilities [1,20]. In glassy systems, aging is also found in the related response functions $R(t,t_w)$, associated with a breakdown of the fluctuation dissipation theorem which at equilibrium would be $TR(t,t_w) = \partial_{t_w} C(t,t_w)$. This is taken into account by introducing an effective temperature $T_{\rm eff}$ through [20]

$$T_{\rm eff}(q) = \lim_{t_w\to\infty} \left.\frac{\frac{\partial C(t,t_w)}{\partial t w}}{R(t,t_w)}\right|_{C(t,t_w)=q}. \qquad (4.1)$$

In coarsening systems, however, response functions have been shown numerically and analytically to be weak, in the sense that $T_{\rm eff} \to \infty$ at large times [21,22]. This property has been related to the decreasing density of topological defects (domain walls) during the coarsening. In the case of the $O(n)$ model, no topological defects are present if $n > d$, which is naturally the case in the large-n limit. We compute then $R(t,t_w)$ in the both cases studied above to obtain $T_{\rm eff}$. We refer the reader to reference [22] for the method, since we follow exactly the same steps. We get the two following expressions:

$$R(t,t_w) \sim \left(\frac{t}{t_w}\right)^{d/4}\left(\frac{1}{t-t_w}\right)^{d/2}, \qquad (4.2)$$

in the non-conserved case, and

$$R(t,t_w) \sim \frac{1}{(t-t_w)^{(d+2)/4}} \times \exp\left(\frac{d}{8}\frac{\left(\sqrt{t\ln t}-\sqrt{t_w\ln t_w}\right)^2}{t-t_w}\right), \qquad (4.3)$$

in the conserved case (we dropped out all numerical constants). Combining equations (2.5, 3.2, 4.2, 4.3), it is easy to show that for the non-conserved and the conserved case successively, one has:

$$T_{\rm eff}(q) \sim \lim_{t_w\to\infty} t_w^{d/2-1},$$
$$T_{\rm eff}(q) \sim \lim_{t_w\to\infty} \frac{t_w^{(d-2)/4}}{(\ln t_w)^{(d+2)/8}\exp(\sqrt{\ln t_w})}\cdot \qquad (4.4)$$

This holds for $0 < q < 1$, and shows that for $d > 2$, although there is no interpretation here in terms of defects, the effective temperature is infinite, as has been found so far in all domain growth processes [4,21,22].

5 Conclusion

We studied in this paper the aging dynamics of the $O(n)$ model in the large-n limit. We recalled first the case of a non conserved order parameter where standard scaling laws hold, leading to a simple t/t_w-aging behaviour [1,2]. We investigated the more interesting case of a conserved dynamics, and were able to show that the multiscaling observed in the structure factor does not imply a hierarchy of time scales ('ultrametricity in time' [18]). Rather, the relaxation takes place in a time scale which is shorter than the waiting time, $t_r \sim t_w/\ln^{a-1}(t_w)$ with $a = 3/2$, the correlation function being well represented in that regime by $C(t,t_w) = \mathcal{C}(h(t)/h(t_w))$, where h is an enhanced power law $h(t) = \exp(\ln^a(t))$. This simple example exhibits then an interesting sub-aging behaviour, whose origin is the presence of two different length scales during the coarsening process. It shows also that the interpretation of $h(t)$ as a length scale may in some cases be misleading. The enhanced exponential form that has been successfully used to fit spin glass experiments arises naturally from our computation. It implies that the relaxation time scales as $t_r \sim t_w/\ln^{a-1}(t_w)$. Experimentally, this scaling form should be hardly distinguishable from a power law $t_r \sim t_w^\mu$, if μ is very near to one, as it seems to be in spin glasses [6].

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